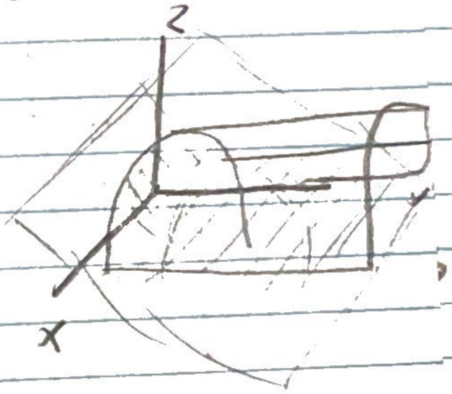
Notes

Recall: __Divergence Theorem__

If R is "nice" and ∂R is also nice and F is a vf on $\mathbb{R}^3$ w/ its partial derivatives, then

$$\iint_{\partial R} F \cdot dS = \iiint_R \operatorname{div}(F)\, dV$$

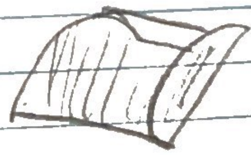
__Ex__: Compute the flux of $F = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$ across S the surface of the region bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, $y = 2 - z$



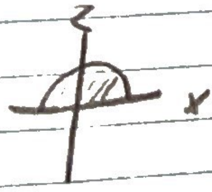
__Sol__: Try to apply Div Thm

$$\operatorname{div}(F) = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[y^2 + e^{xz^2}] + \frac{\partial}{\partial z}[\sin(xy)]$$

$$= y + 2y + 0 = 3y$$



$$R = \left\{ (x, y, z) : \begin{array}{l} -1 \le x \le 1 \\ 0 \le z \le 1 - x^2 \\ 0 \le y \le 2 - z \end{array} \right.$$



$$\therefore \iint_S F \cdot dS = \iint_{\partial R} F \cdot dS = \iiint \operatorname{div}(F)\, dV$$

$$= 3 \int_{x=-1}^{1} \int_{z=0}^{1-x^2} \int_{y=0}^{2-z} y \, dy\, dz\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} \int_{0}^{1-x^2} \left[y^2 \right]_{y=0}^{2-z} dz\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} \int_{0}^{1-x^2} (2-z)^2 - 0 \, dz\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} -\frac{1}{3} \Big[(2-z)^3 \Big]_{0}^{1-x^2} dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left((2-(1-x^2))^3 - (2-0)^3 \right) dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left((1+x^2)^3 - 8 \right) dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left(3x^2 + 3x^4 + x^6 - 7 \right) dx$$

$$= -\frac{1}{2} \left[-7x + x^3 + \frac{3}{5} x^5 + \frac{1}{7} x^7 \right]_{-1}^{1}$$

$$= -\frac{2}{2} \left(-7 + 1 + \frac{3}{5} + \frac{1}{7} \right) \quad \blacksquare$$

Ex: Compute the flux of $F = \langle xye^z, xy^2z^3, -ye^z \rangle$ across the box bounded by the coor planes and $x = 3$, $y = 2$, $z = 1$

Sol: $R = [0,3] \times [0,2] \times [0,1]$

Try to apply div thm

$\text{div}(F) = ye^z + 2xyz^3 - ye^z = 2xyz^3$

$\therefore \iint_{\partial R} F \cdot dS = \iiint_R \text{div}(F)\, dV = \rightarrow$

$2\iiint_R xyz^3\,dV$

$f(x)\cdot g(y)\cdot h(z)$

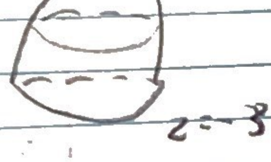
$$= 2\underbrace{\int_{x=0}^{3}\int_{y=0}^{2}\int_{z=0}^{1}}_{\text{constant}} xyz^3\,dz\,dy\,dx$$

$$= 2\left(\int_0^3 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^1 z^3\,dz\right)$$

$$= 2\left[\tfrac{1}{2}x^2\right]_0^3\left[\tfrac{1}{2}y^2\right]_0^2\left[\tfrac{1}{4}z^4\right]$$

$$= \tfrac{1}{8}(3^2)(2^2)(1) = \tfrac{9}{2} \quad \square$$

Ex: Compute flux of $F = \langle 2x^3+y^3,\ y^3+z^2,\ 3y^2z\rangle$ across surface of region bounded by paraboloid $z = 1-x^2-y^2$ and plane $z=-3$



Sol: Try to apply div thm

$\operatorname{div}(F) = 6x^2 + 3y^2 + 3y^2$
$= 6(x^2+y^2)$

shadow

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$R_{cyl} = \{(r,\theta,z) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1-r^2\}$

$-3 = z = 1-x^2-y^2$

$x^2+y^2 = 4$

$$\iint_{\partial R} F\cdot dS = \iiint_R \mathrm{div}(F)\,dV \quad ,$$

div(F)

$$= \iiint_{R(x)} \mathrm{div}(F)_{(x)}\, r\, dV_{(x)}$$

$$= \int_0^{2\pi}\int_{r=0}^{2}\int_{z=-3}^{1-r^2} 6r^2 \, dz\, dr\, d\theta$$

$$= 6\int_0^{2\pi}\int_0^{2} \left[zr^3\right]_{-3}^{1-r^2} dr\, d\theta$$

$$= 6\int_0^{2\pi}\int_0^{2} r(r^2)\left(1-r^2-(-3)\right) dr\, d\theta$$

$$= 6\int_0^{2\pi}\int_0^{2} \left(4r^3 - r^5\right) dr\, d\theta$$

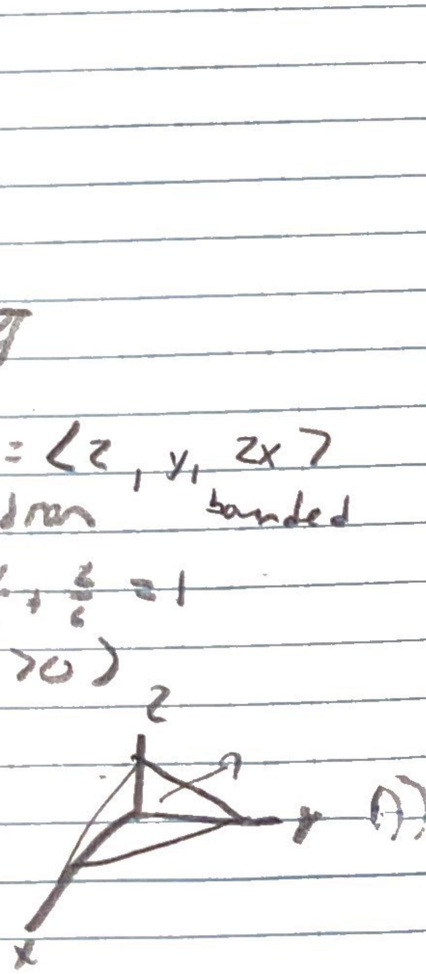
$$= 6\int_0^{2\pi} \left[r^4 - \frac{1}{6}r^6\right]_0^2 d\theta$$

$$= 6\int_0^{2\pi} \left(16 - \frac{32}{3} - 0\right) d\theta$$

$$= 32(2\pi - 0) = 64\pi \quad \blacksquare$$

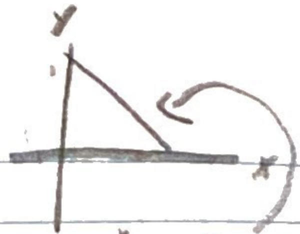
Ex: Compute the flux of $F = \langle z, y, zx \rangle$ across surface of tetrahedron bounded by the coord planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (for unknown constants $a, b, c > 0$)

$n \cdot (x - p) = 0$



$n \cdot x = n \cdot p = 0$

$n \cdot x = d \qquad n = \langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \rangle$



$R = \{(x,y,z) : 0 \le x \le a,\ 0 \le y \le b(1-\frac{x}{a}),\ \frac{x}{a} + \frac{y}{b} = 1$
$0 \le z \le c(1 - \frac{x}{a} - \frac{y}{b})$

$\therefore$ Applying div thm

$div(F) = 1 + x$

$$\therefore \iint_{\partial R} F \cdot dS = \iiint_R div(F)\, dV$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\, dz\, dy\, dx$$

$$= \int_0^a (1+x) \int_0^{b(1-\frac{x}{a})} [z]_0^{1-\frac{x}{a}-\frac{y}{b}}\, dy\, dx$$

$$= \int_0^a (1-x) \int_0^{b(1-\frac{x}{a})} (c(1-\frac{x}{a}-\frac{y}{b})-0)\, dy\, dx$$

$$= c\int_0^a (1+x)\left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a (1+x)(b(1-\frac{x}{a})(1-\frac{x}{a} - \frac{b}{2b}(1-\frac{x}{a}))-0$$

$$= bc\int_0^a (1+x)(1-\frac{x}{a})(1-\frac{x}{a})\, dx$$

$$= \frac{1}{2} bc \int_0^a (1 + (1 - \frac{1}{a} - \frac{1}{a})x + (-\frac{1}{a} - \frac{1}{a} + \frac{1}{a^2})x^2 + \frac{1}{a^2}x^3\, dx$$

$$= \frac{1}{2} bc \int_0^a \left(1 + \left(1 - \frac{2}{a}\right) x + \left(\frac{1}{a^2} - \frac{2}{a}\right) x^2 + \frac{1}{a^4} x^3\right) dx$$

$$= \frac{1}{2} bc \left[x + \frac{1}{2}\left(1 - \frac{2}{a}\right) + \frac{1}{3}\left(\frac{1}{a^2} - \frac{2}{a}\right) x^3 + \frac{1}{4a^4} x^4 \right]_0^a$$

$$= \frac{1}{2} bc \left(a + \frac{1}{2}\left(1 - \frac{2}{a}\right) a^2 + \frac{1}{3}\left(\frac{1}{a^2} - \frac{2}{a}\right) a^3 + \frac{a^4}{4a^4} \right)$$

$$= \frac{1}{2} abc \left(1 + \frac{1}{2}(a-2) + \frac{1}{3}(1-2a) + \frac{1}{4a} \right)$$

$$= \frac{1}{2} abc \left(\frac{1}{3} + a\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)\right) = \frac{1}{6} abc \left(1 + \left(\frac{1}{4}\right) a\right)$$

$$= \frac{1}{6} abc \left(1 - \frac{1}{4a}\right) \quad \square$$